The Viewing Graph

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Abstract

The problem we study is; Given N views and a subset of the $\binom{N}{2}$ interview fundamental matrices, which of the other fundamental matrices can we compute using only the pre-computed fundamental matrices. This has applications in 3-D reconstruction and when we want to reproject an area of one view on another, or to compute epipolar lines when the correspondence problem is too difficult to compute between every two views.

A complete solution using linear algorithms to compute the missing fundamental matrices is given for up to six views. In many cases problems with more than six views can also be handled.

Keywords: *Multi-image Structure, Projective Reconstruction, Structure From Motion, Multiple View Geometry, Linear Reconstruction Techniques.*

1. Introduction

Two-view geometry has been explored since 1980, in 1981 the fundamental matrix was introduced to the computer vision community by Longuet-Higgins [6]. Epipolar relations were discovered and three-view geometry was explored till the trifocal tensor was discovered [1, 2, 10]. Four-view geometry was studied resulting in the discovery of the quadrifocal tensor in 1995 [3, 4].

These tensors contain many non-linear dependencies (the trifocal tensor has 27 entries but just 18 d.o.f. and the quadrifocal tensor has 81 entries and just 29 d.o.f) which makes its geometric understanding harder [5, p. 378,411-413]. There is no single tensor that can describe the N-view geometry for $N > 4$. These two reasons lead us to explore the general N view problems with the most basic mathematical entity, the fundamental matrix, which fully describes the relation between two views with just one non linear relation between it's components. The fundamental matrix can be computed linearly from 8 point correspondences between two views, and non linearly from 7 points [5, chap. 10].

An N-view scene can be represented as a graph whose nodes are the views and whose edges are the known fundamental matrices between the views. The problem we study is; Given N views and a subset of the $\binom{N}{2}$ edges (fundamental matrices) what other edges (fundamental matrices) can we compute only using the values of the given edges.

The need to solve such a question arises in 3-D reconstruction and when we want to reproject an area of one view on another, or to compute epipolar lines when the correspondence problem is too difficult to compute between every two views.

We are mainly interested in *solving graphs*, graphs such that all of whose missing edges can be computed. We present a complete description of these graphs of up to 6 views in general position, and an algorithm to compute the missing fundamental matrices. The tools that we present also enables one to compute edges in many larger graphs only solving linear equations. An anonymous referee refered us to an unpublished manuscript of J. Goldberger who proposed an algorithm for the general case that works in certain cases.

2. The four-view case

Denote the fundamental matrix between view i and view j by $F_{i,j}$ (passes points from view j to epipolar lines in view i), and by e_{ij} the epipole in view i of camera center j.

Theorem 2.1 *[Hartley Zisserman [5, p. 238]] A pair of camera matrices corresponding to a fundamental matrix $F_{1,0}$ is given by $C_0 = (I; 0)$ and $C_1 = ([e_{1,0}]_\times F_{1,0}; 0) + e_{1,0}x^t$ where $e_{1,0}$ is the epipole satisfying $e_{1,0}^t F_{1,0} = 0$, and x is an arbitrary 4-vector ($x_4 \neq 0$).*

Theorem 2.2 *[Hartley Zisserman [5, p. 400]] If A, B are camera matrices, then- $F_{B,A}[j,i] = (-1)^{i+j} det \begin{bmatrix} \sim A^i \\ \sim B^j \end{bmatrix}$ where $\sim C^i$ is the matrix obtained from camera matrix C by omitting row i.*

Lemma 2.3 *A solving graph with more than 2 views cannot contain a view with only one neighbor.*

Proof: By contradiction, denote the view with only one neighbor by C_1 its only neighbor by C_0, $F_{1,0}$ the fundamental matrix between them, $e_{1,0}$ its epipole, and a third view in the graph by C_2. By a change of coordinates in the world, take $C_0 = (I;0)$, then by theorem 2.1, $C_1 = ([e_{1,0}]_\times F_{1,0};0) + e_{1,0}x^t$ where x is the 4 d.o.f. then by theorem 2.2 with cameras C_2 and C_1 we see that F has the same 4 d.o.f. hence, the graph is not a solving graph. ■

From Lemma 2.3 it follows that any solving 4 graph includes the one described in figure 1 where an edge signifies a known fundamental matrix.

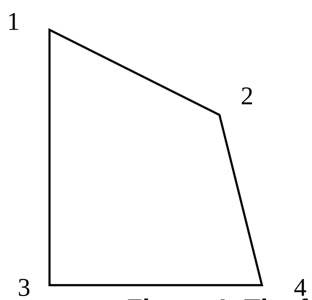


Figure 1. The four view graph.

What can we say about $F_{2,3}$ or $F_{1,4}$?

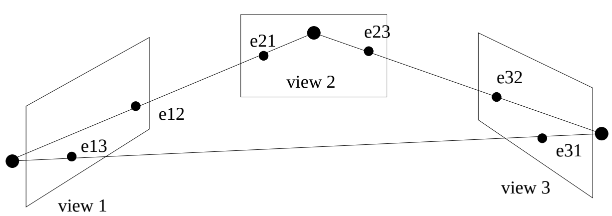


Figure 2. Three views and epipoles.

It is known that:

Theorem 2.4 *Let 1,2,3 be three views with centers not co-linear. Denote by $e_{i,j}$ the epipole in view i of view j, then $F_{1,2}e_{2,3} \cong e_{1,2} \times e_{1,3}$.*

Proof: $F_{1,2}e_{2,3}$ is the projection of the viewing line of $e_{2,3}$ (in view 2) onto view 1 (see figure 2). It is the projection of the baseline of views 2 and 3 -the line between their camera centers, hence it is the line in view 1 passing through $e_{1,2}$ and $e_{1,3}$, $e_{1,2} \times e_{1,3}$. ■

Theorem 2.5 *All possible fundamental matrices $F_{2,3}$ that complete a triangle with given $F_{1,2}$ and $F_{1,3}$ are a 4-dimensional linear subspace of P^8.*

Proof: By a change of coordinate frame in the world we take C_1 as $(I;0)$ and $C_3 = ([e_{3,1}]_\times F_{3,1}; e_{3,1}$. $C_2 = ([e_{2,1}]_\times F_{2,1};0) + e_{2,1}x^t$ (see theorem 2.1). Using theorem 2.2 we see that all the non linear terms in $F_{2,3}$ vanish and we are left with a four dimensional projective space where the x vector is the d.o.f. (We know that these d.o.f. do not vanish, because all three cameras with three non co-linear centers are equivalent, upto 2-D homographies of the images, and give the same space which is four dimensional as can be checked on a numeric example). ■

Theorem 2.6 *Given the fundamental matrices $F_{1,2}, F_{2,4}, F_{4,3}$ and $F_{3,1}$ between views 1-4, where the camera centers are not coplanar (see figure 1) :*

a. $F_{2,3}$ is a member of a line of fundamental matrices (the same with $F_{1,4}$).

b. The epipoles $e_{2,3}$ and $e_{3,2}$ can be computed (the same with $e_{1,4}$, $e_{4,1}$).

Proof: We use theorem 2.4 several times to get the following equations:

$$F_{3,4}e_{4,2} \cong e_{3,2} \times e_{3,4} \tag{1}$$

$$F_{3,1}e_{1,2} \cong e_{3,2} \times e_{3,1} \tag{2}$$

1 and 2 yield[1]:

$$e_{3,2} \cong F_{3,4}e_{4,2} \times F_{3,1}e_{1,2} \tag{3}$$

Similarly:

$$e_{2,3} \cong F_{2,4}e_{4,3} \times F_{2,1}e_{1,3} \tag{4}$$

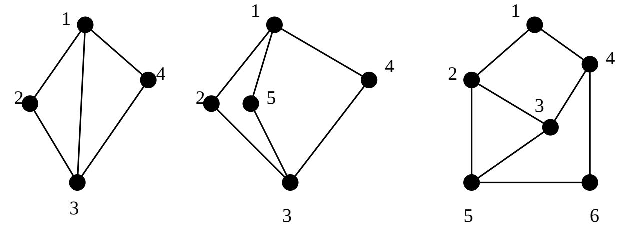
In equations 3 and 4 the four fundamental matrices are given and the epipoles in the right side can be computed as their null spaces ($F_{i,j} = (F_{j,i})^t$ and $F_{i,j}e_{ji} = 0$). So we get 2.6(b). $F_{1,4}$ lies in the intersection of the two projective 4 dimentional spaces indicated in theorem 2.5 about $F_{3,1}$ and $F_{3,4}$ and about $F_{2,1}$ and $F_{2,4}$. Since any 4 cameras with non coplanar centers are equivalent up to a choice of coordinate systems in the views, a numeric[2] check shows that in the generic case $F_{1,4}$ is a member of one dimensional linear projective space in P^8. Hence 2.6(a). ■

Definition 1 *The graph in figure 3(a) will be called 4g5 (4 graph with 5 edges), the graph in figure 3(b), 5g6, and the graph in figure 3(c), 6g8.*

Later on we see that these are the minimal solving graphs with up to 6 views.

[1] Here I assume the 4 camera centers are not coplanar.

[2] Using strong enough symbolic software one can directly verify that this intersection is a one dimensional projective space.



(a) 4g5 (b) 5g6 (c) 6g8

Figure 3. The three most simple solving graphs.

Theorem 2.7 *If P_1 P_2 P_3 and P_1 $P_4 P_3$ (the camera centers in figure 3(a)) are not colinear then 4g5 is a solving graph.*

Proof: The fundamental matrices $F_{1,2}, F_{1,4}, F_{4,3}, F_{2,3}$and $F_{3,1}$are given. By theorem 2.1 cameras C_2 and C_3 can be chosen arbitrarily as $C_1 := (I;0)$ and $C_3 := ([e_{3,1}]_\times F_{3,1};0) + e_{3,1}x^t$ where x is an arbitrary 4-vector.

Taking $P \in P^4$ a point in the world, $p_1 \cong C_1 P$ and $p_3 \cong C_3 P$ are its images in views 1 and 3. $F_{4,1}p_1$ and $F_{4,3}p_3$ are the epipolar lines passing in view 4 through its image- p_4, so $p_4 \cong C_4 P \cong F_{4,1}p_1 \times F_{4,3}p_3$ (Since P_1 P_4 and P_3 are not colinear, the epipolar lines $F_{4,1}p_1$ and $F_{4,3}p_3$ are not always the same). Doing so 6 times gives enough linear equations to recover C_4. In the same way we recover C_2, and by theorem 2.2 we get $F_{2,4}$. ■

Thus the graphs with 5 or 6 edges are the only solving graphs with 4 nodes.

3. Five-view graphs

Theorem 3.1 *When 5 views are in general position*[3] *then 5g6 is a solving graph.*

Proof: In figure 3(b) using theorem 2.6(a) once with views 1,2,3 and 4, and then with views 1,2,3 and 5 both looking at the lines which contain $F_{1,3}$, these two lines (in P^8 - the projective 8 dimensional space) meet in a single fundamental matrix ($F_{1,3}$. Then, using theorem 2.7 we can compute $F_{2,5}$ and $F_{5,4}$. This structure will not be unique iff the two lines in P^8 coincide. ■

If G_1, G_2, G_3 and G_4 are any graphs, constructed as shown in figure 4(a), the 1 d.o.f between views 2 and 3 of theorem 2.6(a) will stay. Figure 4(b), 4(c) and 4(d) are special cases of this argument, and by the same argument stated in the proof of lemma 2.3 we get that views 1&2 in figure 4(e) have the same 4 d.o.f. so it is not a solving graph either, so we have:

[3]The centers $P_1 P_2 P_3 P_5$ and $P_1 P_5 P_3 P_4$ are not coplanar and the two lines indicated in the proof do not coincide.

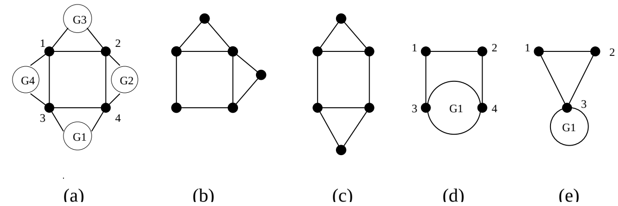


Figure 4. Non solving graphs.

Lemma 3.2 *The graphs in figure 4 a-e are not solving graphs.*

The fact that the graphs in figure 4(d) and (e) are not solving graphs indicates that a necessary condition for a graph to be a solving graph is that there are no 2 adjacent vertices of degree 2. Together with lemma 2.3 we get a necessary condition for a solving graph*:*

Corollary 3.3 *A solving graph with more than 3 views must satisfy the following condition: all the vertices are of degree at least 2, and it has no two adjacent vertices of degree 2.*

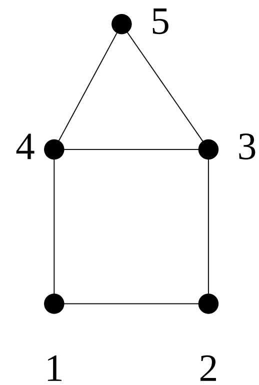


Figure 5.

The following theorem shows that using only the necessary condition we can recognize all the 5 view solving graphs:

Theorem 3.4 *A 5-view graph is a solving graph iff it satisfies the necessary condition i.e. every vertex is of degree at least 2 and there are no two neighbors with degree 2.*

Proof: If the graph is a solving graph, corollary 3.3 says it satisfies the necessary condition. If the graph satisfies the necessary condition, first, we show it must have a 4-cycle: It must have a vertex of degree at least 3, denote it by *a*, and *a*'s neighbors by *b*,*c* and *d*. The necessary condition implies that the fifth vertex (*e)* has at least 2 edges with *a*-*d*. If the 2 edges are with b, c or d, there is a 4-cycle, if one edge is with *a*, the second w.l.o.g. is with *d*. The necessary condition implies that vertex c or b has one more edge with d or e so again there is a 4-cycle. Denote the 4-cycle's vertices 1-4. The fifth vertex must have 2 edges adjacent to this 4-cycle so either a 5g6 is constructed and we are done, or a graph such as in figure 5, but then the necessary condition implies that vertex 1 or 2 will have one more edge and a 4g5 is constructed. In the two cases using theorem 2.7 several times solves the graph. ■

4. Six-view graphs

In this section we show that theorems 2.7, 3.1 and 4.1 enable us to determine all 6-view solving graphs.

We use the fact that any time we find a 4g5 or 5g6 in the graph, the necessary condition implies that one of the other vertices has 2 edges connected to it, so using theorem 2.7 several times solves the graph.

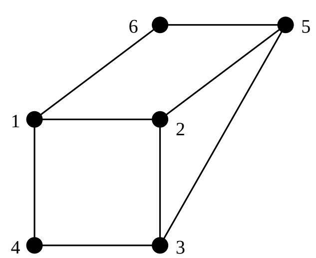


Figure 6. The graph 6g8.

Theorem 4.1 *6g8 (see figure 6) is a solving graph when the six views are in general position*[4].

Proof: In figure 6 by theorem 2.6(a) on views 1,2,3 and 4, $F_{1,3}$ is a linear function of one variable x, $F_{1,3}(x)$. Theorem 2.6(b) enables us to calculate $e_{1,3}$. Looking at views 1,2,5 and 6 with theorem 2.6(b) we calculate $e_{1,5}$. $e_{3,5}$ is the null space of $\mathrm{F}_{5,3}$, and using theorem 2.4 we have:

$$F_{1,3}(x)e_{3,5} = e_{1,3} \times e_{1,5} \tag{5}$$

which enables us to find x or $F_{1,3}(x)$ linearly, and we have 4g5 which enables us to solve the graph as mentioned earlier. ■

Theorem 4.2 *A 6-view graph is a solving graph iff it satisfies the necessary condition and contains one of 4g5, 5g6 or 6g8.*

Proof: If the graph is a solving graph, by 3.3 we get that it satisfies the necessary condition.

Similarly to the previous section, a 6-view graph satisfying the necessary condition must contain a 4-cycle. We

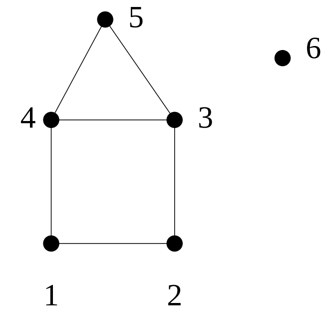


Figure 7.

denote its vertices by 1 to 4, (see figure 7). The necessary condition implies that one of vertices 5 or 6 has 2 edges with vertices 1 to 4 (w.l.o.g. vertex 5). If the edges are with vertices 2 & 4 or with 1 & 3 we get a 5g6. Otherwise, we can assume that vertex 5 is connected with 3 & 4. Vertex 6 has 2 edges. If the edges are with vertices 4 & 5 or with 3 & 5 we get a 4g5 and if with 1 & 3 or 2 & 4 we get a 5g6.

All the other possibilities (1&5 or 2&5 or 2&3 or 2&1 or 1&4) create one of the three graphs listed in figure 8.

By corollary 3.2 the graphs in figures 8(a) and 8(b) are not solving graphs so we are left just with the graph in figure 8(c) which is 6g8.

The second direction: If the graph contains 6g8 by 3.3 the graph is solved. If it contains 5g6 by 3.1 this subgraph is solved, and the necessary condition implies that the 6'th vertex (which is not in the 5g6) has two edges with it. Any edge of the kind (6,i) can be calculated using 2.7. If the graph contains 4g5, by theorem 2.7 this subgraph is solved and the necessary condition implies that one of vertices 5 or 6, say 5 (which are not in the subgraph) has 2 edges with it, so again using 2.7 we find all edges of the form (i,5). Using the same argument with vertex 6 solves the graph. ■

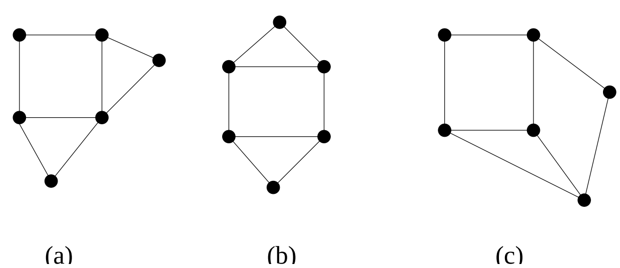


Figure 8. Two non solving graphs and 6g8

5. Some bigger solving graphs

With the tools developed till now many bigger graphs ($N > 6$) can be detected as solving graphs. Hereby few examples of such graphs:

[4] Here we exactly need camera centers $\mathrm{P}_1\mathrm{P}_2\mathrm{P}_3\mathrm{P}_4$ and $\mathrm{P}_1\mathrm{P}_2\mathrm{P}_5\mathrm{P}_6$ are not coplanar and $\mathrm{P}_1\mathrm{P}_3\mathrm{P}_5$ are not colinear.

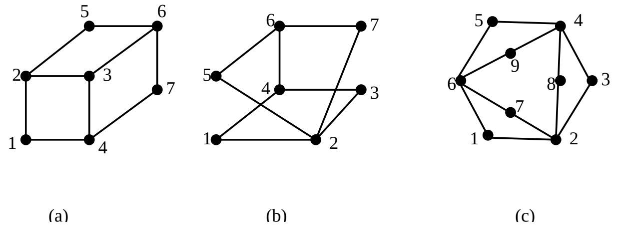


Figure 9. Solving graphs with more than 6 views.

In 9(a) using theorem 2.6(a) we can compute the linear one parameter (x) family of possible fundamental matrices $F_{4,2}(x)$, between views 4 and 2. Using 2.4 we get the relation $F_{4,2}(x) * e_{2,6} = e_{4,2} \times e_{4,6}$, where $e_{4,2}$ $e_{2,6}$ and $e_{4,6}$ can be found using 2.6(b). Thus $F_{4,2}$ can be solved from linear equations, and in the same manner all the other fundamental matrices.

In 9(b) using the same arguments we solve the linear equation $F_{2,6}(x) * e_{6,4} = e_{2,6} \times e_{2,4}$ and get $F_{2,6}$ and in the same way we get $F_{4,2}$. Using theorem 2.7 several times we find all the other fundamental matrices.

In 9(c) in the same way we solve the linear equation $F_{2,6}(x) * e_{6,4} = e_{2,6} \times e_{2,4}$ and get $F_{2,6}$. In the same way we get $F_{6,4}$ and $F_{4,2}$, and by using theorem 2.7 we get the other fundamental matrices.

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