Large Cardinals, Inner Models and Determinacy

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Large Cardinals

Recapitulation:

We have seen that a measure U on $\mathcal{P}(\kappa)$ in V yields an ultrapower $(\mathrm{Ult}(V, U), E)$ which is wellfounded and hence isomorphic to a transitive inner model $(M, \in)$ of ZF. The following facts hold:

- $V_{\kappa+1} = (V_{\kappa+1})^M$
- $(j(\kappa)\text{is measurable})^M$;
- $U \notin M$ and thus $V_{\kappa+2} \neq (V_{\kappa+2})^M$.
- κ may, or may not, be measurable in M (*via* some other measure $\overline{U}$). (If κ was the least measurable of V then it will not, by the Scott argument).

• Modern set theory now classifies large cardinals (that previously were often argued for "by analogy with ω" or some such), into a hierarchy given by the embedding properties that they enjoy.

Definition
A cardinal κ is α-strong if there is an embedding $j : V \longrightarrow_e M$ with $V_a = (V_\alpha)^M$ with $\mathrm{cp}(j) = \kappa$, and $j(\kappa) \geq \alpha$.

• Thus a measurable cardinal is $\kappa + 1$ strong. The larger the α, the stronger the embedding, as more of the initial *V* hierarchy is preserved.

Definition
A cardinal κ is strong *if it α-strong for all α.*

• NB: the order of quantifiers: for every α there is an embedding *j* (depending on α ...).

Extenders

In order to give a first order formalisation of such embeddings: it is possible to give an *extender representation* of such embeddings.

Given an α-strong embedding $j: V \longrightarrow N$ we derive an α-*extender at* κ generalising what we did for measures.

$$X \in E_a \leftrightarrow_{\text{df}} X \in \mathcal{P}([\kappa]^{|a|}) \wedge a \in j(X).$$

The sequence $\mathcal{E} = \langle E_a : a \in [\alpha]^\omega \rangle$ then has satisfactory coherence properties, in fact enough so that we can define an *extender ultrapower* $\text{Ult}(V, \mathcal{E})$ from it. In the situation described, this ultrapower has a wellfounded *E*-relation, and is again isomorphic to some $(M, \in)$ and we shall have again [Diag]

• It is possible to view the $\text{Ult}(V, \mathcal{E})$ as a direct limit of the 'ordinary' ultrapowers by the measures E_a. It is part of the flexibility of the approach that this is inessential though.

Having thus generalised the notion of measure ultrapower to extender ultrapower we can use these to give us first order formulations of α-strong etc. A simplified statement is:

Lemma

Let α be a strong limit cardinal; then κ is α-strong iff there is an α-extender sequence $\mathcal{E} = \langle E_a : a \in [\alpha]^{<\omega} \rangle$ at κ, with $V_{\kappa+\alpha} \subseteq \text{Ult}(V, \mathcal{E}) \land j(\kappa) > \alpha$.

• The notion of κ being strong is then also first order (although involving a quantifier over On).

• κ strong implies that $V_\kappa \prec_{\Sigma_2} V$.

Definition

κ is *superstrong* if there is $j : V \longrightarrow_e M$ with $V_{j(\kappa)} \subseteq M$.

• α-strong only asked for $V_\alpha \subseteq M$ whilst $j(\kappa) > \alpha$. This seemingly innocuous extension is in fact a powerful strengthening. Again it has a first order formalisation.

Towards Woodin cardinals

We proceed to a definition of Woodin cardinal. First we define a strengthening of the concept of strong.

Definition
Let $A \subseteq V$. We say that κ is A-strong in V if for every α there are M, $B \subseteq V$ with $IM(M)$ and an $\mathcal{L}_{\dot{\in}, \dot{A}}$-elementary embedding

$$j : \langle V, A \rangle \longrightarrow_e \langle M, B \rangle$$

such that $\mathrm{cp}(j) = \kappa$, $V_\alpha \subseteq M$ and $V_\alpha \cap A = V_\alpha \cap B$.

This is not a first order formalisation, but now consider an inaccessible λ and relativise the notion from V down to V_λ:

Definition
An inaccessible cardinal λ is called Woodin *if for every $A \subseteq V_\lambda$ there is a $\kappa < \lambda$ which is A-strong in V_λ.*

• A Woodin cardinal is necessarily Mahlo, but may fail to be weakly compact.

Lemma
(i) If κ is superstrong then it is a Woodin limit of Woodins.
(ii) If λ is Woodin then ("*there are arbitrarily large strong cardinals*")$^{V_\lambda}$

• A particular constellation of cardinals is also of interest for determinacy of infinite games played with reals, rather than integers. The assertion "$AD^{\mathbb{R}}$" is that for every $\mathcal{A} \subseteq {}^\omega\mathbb{R}$ the relevant game $G_{\mathcal{A}}$ is determined.

Conjecture (*The "$AD^{\mathbb{R}}$ hypothesis*") The consistency strength of $AD^{\mathbb{R}}$ is that of a cardinal μ that is a limit of infinitely many Woodins $\lambda_n < \lambda_{n+1} \cdots < \mu$ but also of of μ-strong cardinals $\kappa_n < \lambda_n < \kappa_{n+1}$.

Supercompact Cardinals

• We continue our cataloguing of some more large cardinals through elementary embeddings.

Definition
(i) A cardinal κ is α-supercompact if there is a $j : V \longrightarrow_e M$ with $^\alpha M \subseteq M$.
(ii) κ is supercompact if it is α-supercompact for all α.

• This is also a central notion: there are many theorems, especially in forcing arguments, that assume the consistency of supercompact cardinals.

Extendible Cardinals

Definition (Silver, after Reinhart)

*(i) A cardinal κ is α-extendible if there are λ, j with $j : V_{\kappa+\alpha} \longrightarrow_e V_{\lambda+\alpha}$
$\wedge \operatorname{cp}(j) = \kappa$.*
(ii) κ is extendible if it is α-extendible for all α.
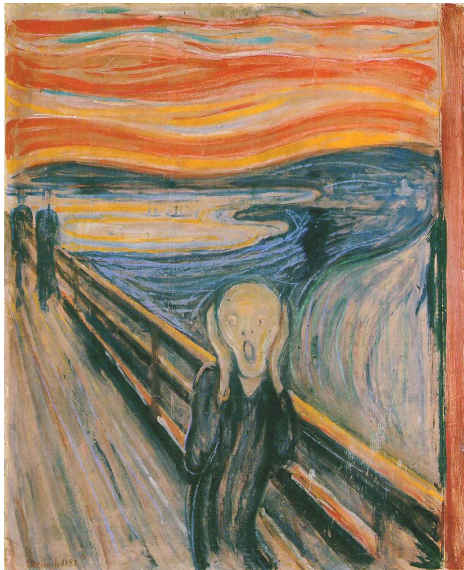
1-extendibility is a strong concept:

Lemma

If κ is 1-extendible, then it is superstrong (and there are many such below it.)

Lemma

If κ is extendible, it is supercompact; κ extendible implies $V_\kappa \prec_{\Sigma_3} V$.

Towards The Unknown Region:

However if we try to maximise the extendibility properties we run into inconsistency:

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Theorem (Kunen)

$(\text{ZFC}_{\dot{j}})$ *There is no non-trivial $\mathcal{L}_{\dot{\in}}$-elementary embedding $j : V \longrightarrow_e V$.*

It is unknown whether AC is necessary for this theorem. The proof actually yields a direct ZFC result:

Theorem (Kunen)

(ZFC) *There is no non-trivial elementary embedding $j : V_{\lambda+2} \longrightarrow_e V_{\lambda+2}$.*

The "2" is an essential artefact of the argument. That there may be a non-trivial $j : V_{\lambda+1} \longrightarrow_e V_{\lambda+1}$ is not known to be inconsistent; if this is to be the case, then $\kappa_0 = \text{cp}(j) < \lambda$ and it can be shown that λ has cofinality ω being $\sup\{\kappa_0, j(\kappa_0), jj(k_0), \dots\}$.

This is not the end of the story however, as for such $V_{\lambda+1}$ the model $L(V_{\lambda+1})$ and its putative elementary embeddings has become an object of intense study.